Consider Two sets $A=\{1,2,3\}$, $B=\{4,5\}$

Then $A\times B=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

Any sub set of $A\times B$ is called a relation from set A to set B.

i.e $R: A \to B$

Empty relation: A relation R in a set A is called empty relation, if No element of set A is related to any element of set A

i.e $R=\phi \subseteq A\times A$.

Universal relation: A relation R in a set A is called universal relation, if each element of set A is related to every element of set A

i.e $R = A\times A$

Both the empty relation and universal relation are some times called Trivial relation.

Types of relations.

i Reflexive relation:- A relation R in a set A is called reflexive, iff $(a,a)\in R$ for every $a\in A$.

ii A relation R in a set A is called symmetric iff $(a,b)\in R \Rightarrow (b,a)\in R$ for all $a,b\in A$

iii A relation R in a set A is called transitive iff $(a,b)\in R$ & $(b,c)\in R \Rightarrow (a,c)\in R$ for all $a,b,c\in A$.

A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

EXE. 1.1 Q1. Determine whether each of the following relation are reflexive, symmetric and transitive.

(a) Relation R in the set $A=\{1,2,3 - - - 13,14\}$ defined as

$R=\{(x,y): 3x-y=0\} = \{(x,y): 3x=y\}$

$=\{(1,3)\ (2,6)\ (3,9)\ (4,12)\}$.

R is not reflexive, $\because (a,a)\notin R$, $\therefore$ R is not symmetric $\because (1,3)\in R$ $(3,1)\notin R$

Also $(1,3)\in R$ & $(3,9)\in R \Rightarrow (1,9)\notin R$ thus R is not transitive

Show that the relation R in the set A of points in a plane given by $R = \{(P, Q) :$ distance of the point P from the origin is same as the distance of the point Q from the origin$\}$, is an equivalence relation. Further, show that the set of all points related to a point $P \neq (0,0)$ is the circle passing through P with origin as centre.

Sol. Let O be the origin. $R = \{(P,Q) : |OP| = |OQ|\}$ where O is origin.

Since $|OP| = |OP| \Rightarrow (P,P) \in R$ for all $P \in A$

$\therefore$ R is reflexive.

Also $(P,Q) \in R \Rightarrow |OP| = |OQ| \Rightarrow |OQ| = |OP| \Rightarrow (Q,P) \in R$

thus R is symmetric.

Now let $(P,Q) \in R$ and $(Q,T) \in R$

$\Rightarrow |OP| = |OQ|$ & $|OQ| = |OT| \Rightarrow |OP| = |OT| \Rightarrow (P,T) \in R$

$\therefore$ R is Transitive

Hence R is an equivalence relation.

Set of points related to $P \neq O$

$\{Q \in A : (Q,P) \in R\} = \{Q \in A : |OQ| = |OP|\}$

$= \{Q \in A : Q$ lies on a circle through P with centre O$\}$

Q. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$ is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related.

Sol. Every triangle is similar to itself. therefore TRT for all $T \in A$

$\therefore$ R is reflexive. Similarly T_1 is similar to T_2 & T_2 is similar to T_1

$(T_1, T_2) \in R \Rightarrow (T_2, T_1) \in R$ Hence R is symmetric.

Also $T_1 \cong T_2$ & $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$ Hence R is Transitive.

i.e $(T_1, T_2) \in R$ & $(T_2, T_3) \in R \Rightarrow (T_1, T_3) \in R$

Hence R is an equivalence relation on set A

Two triangles are similar iff sides are proportional

As $3 : 4 : 5 :: 6 : 8 : 10 \Rightarrow T_1$ is related to T_3. i.e T_1 & T_3 are similar Δs

Q13

B

Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Sol

$A \Rightarrow$ set of all polygons.

$R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$

Since P & P have same number of sides $\therefore (P, P) \in R$ for all $P \in A$

$\therefore$ R is reflexive.

Also $(P_1, P_2) \in R \Rightarrow P_1$ & P_2 have same number of sides

$(P_2, P_1) \in R \Rightarrow P_2$ & P_1 have same number of sides

$\therefore (P_1, P_2) \in R \Rightarrow (P_2, P_1) \in R$ $\therefore$ R is symmetric.

Also $(P_1, P_2) \in R$ & $(P_2, P_3) \in R \Rightarrow P_1$ & P_2 have same number of sides

(P_2, P_3) have same number of sides

$\therefore (P_1$ & $P_3)$ have same number of sides

$\therefore (P_1, P_3) \in R$; Hence R is Transitive

Hence R is an equivalence relation.

Q14 is struck

Q15 B 16 C.

[illegible] has already been explained [illegible] identity function, constant function, [illegible] function [illegible] along with their [illegible]

One-one function

Definition: A function $f: X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise f is called many-one [illegible]

Definition: A function $f: X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$ there exists an element x in X such that $f(x) = y$ [illegible] onto function.

[illegible]

(1) Show that the function $f: R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Sol. It is given that $f: R_* \to R_*$ is defined by $f(x) = \frac{1}{x}$.

To show one-one: $f(x) = f(y) \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y$

$\therefore$ f is one-one function.

To show f is onto: [illegible] $y \in R_*$, there exists $x = \frac{1}{y} \in R_*$ such that $f(x) = \frac{1}{1/y} = y$. Hence f is onto function [illegible]

Hence f is one-one & onto (10)

Now, consider function $g: N \to R$, defined by $g(x) = 1/x$

g is one-one function $\because g(x_1) = g(x_2)$ for every $x_1, x_2 \in X$

$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

Hence g is one-one function.

Further g is not onto as for $1.2 \in R$ but there does not exist any $x \in N$ such that $g(x) = \frac{1}{1.2}$

$\therefore$ g is one-one but not onto function.

Q. 2: Check the injectivity and surjectivity of the following function: $f: N \to N$ given by $f(x) = x^2$

Sol $x_1, x_2 \in N$ such that $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

Hence f is ~~onto~~ one-one function i.e. f is injective.

f is not surjective i.e. onto

$\because$ Range of $f = \{1^2, 2^2, 3^2, \ldots, n^2\} \neq N$

(ii) Sol $x_1, x_2 \in Z$ be such that $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1^2 - x_2^2 = 0 \Rightarrow (x_2 - x_1)(x_2 - x_1) = 0$

$x_1 = x_2$ & $x_1 = -x_2$

$\therefore f(x_1) = f(-x_1)$ for all $x_1 \in Z$

$\therefore$ f is not one-one function i.e. f is not injective

Also Range of $f = \{0^2, 1^2, 2^2, \ldots\} \neq Z$.

$\therefore$ f is not onto function i.e. surjective.

Hence f is not one-one & onto

(iii) $f: R \to R$ defined by $f(x) = x^2$

Sol. $x_1, x_2 \in R$ be such that $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_2^2 - x_1^2 = 0$

$(x_2 + x_1)(x_2 - x_1) = 0 \Rightarrow x_1 = x_2$ & $x_1 = -x_2$

$\therefore f(x_1) = f(-x_1)$ $\therefore$ f is not one-one function

Range of f does contain negative real number $\therefore$ f is not onto.

Q.3 Prove that greatest integer function $f: R \to R$ given by $f(x) = [x]$ is neither one-one nor onto. Where $[x]$ denotes greatest integer function less than or equal to x

Sol For all $x \in [[x], [x]+1)$, $f(x) = [x]$
$\Rightarrow$ $f(x)$ has the same value at infinitely many points
Hence, f is not one-one.
As a particular example for all $x \in [0,1)$ $f(x) = 0$
Also range of f is the set of integers $\neq R$
$\therefore$ f is not onto i.e f is not surjective.

Q4 Show that the modulus function $f: R \to R$ given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$ if x is negative.

Sol. $|x| = \sqrt{x^2} = \sqrt{(-x)^2}$ for all $x \in R$
$\Rightarrow f(x) = f(-x)$ for all $x \in R$, therefore f is not one-one.
Also Range of f contains only non-negative reals i.e range of $f = [0\ \infty) \neq R$. $\therefore$ f is not onto function

Hence f is neither one-one nor onto.

Q5 Show that the signum function $f: R \to R$, given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto

Sol Here $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ $\therefore$ $f(x) = 1$ for all $x \in [0, \infty)$
$f(x) = -1$ for all $x \in (-\infty, 0)$
i.e $f(x)$ has same value at infinitely many points
$\therefore$ f is not one-one function

continued] Also, range of $f = \{-1, 0, 1\} \neq R$

$\therefore f$ is not onto.

Q6. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1,4), (2,5), (3,6)\}$ be a function from A to B. Show that f is one-one.

Sol.

Here $f = \{(1,4), (2,5), (3,6)\}$

$f(1) = 4$, $f(2) = 5$, $f(3) = 6$

Different points of the domain have different f-images in the range

Hence f is one-one

Q7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer

(i) $f: R \to R$ defined by $f(x) = 3 - 4x$

(ii) $f: R \to R$ defined by $f(x) = 1 + x^2$

Sol. (i) Given that $f(x) = 3 - 4x$ for all $x \in R$

Let $x_1, x_2 \in R$ be such that $f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2 \Rightarrow x_1 = x_2 \therefore f$ is one-one

Let $y \in R$ for any real number then $f(x) = y$

$\therefore 3 - 4x = y$ or $-4x = y - 3 \Rightarrow x = \frac{3-y}{4}$

then corresponding to every $y \in R$ there exists $x = \frac{3-y}{4}$

such that $f\left(\frac{3-y}{4}\right) = 3 - \cancel{4} \cdot \frac{3-y}{\cancel{4}} = \cancel{3} - \cancel{3} + y = y$

$\therefore f$ is onto. thus f is bijection

(ii) Let $x_1, x_2 \in R$ be such that $f(x_1) = f(x_2)$

$\therefore 1 + x_1^2 = 1 + x_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow f(x_1) = f(-x_2)$

$\therefore f$ is not one-one function

Also range of f contains only those real which are greater than or equal to 1 $\therefore$ Range of $f \neq R \Rightarrow f$ is not onto.

Q.8 Let A and B be sets. Show that $f: A \times B \to B \times A$ such that $f(a,b) = f(b,a)$ is bijective function.

Sol. Let $(a,b), (a',b') \in A \times B$ be such that

$f(a,b) = f(a',b') \Rightarrow (b,a) = (b',a')$

$\Rightarrow b = b'$ & $a = a' \Rightarrow (a',b') = (a,b)$

Hence f is one-one.

Also corresponding to every ordered pair $(y,x) \in B \times A$ i.e $y \in B$ & $x \in A$ there exists $(x,y) \in A$ such that $f(x,y) = (y,x)$ $\therefore f$ is onto. Hence f is bijective.

Q.9. Let $f: N \to N$ be defined by

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \quad \text{for all } n \in N.$$

State whether the function is bijective. Justify your answer.

Sol.

$f(1) = \frac{1+1}{2} = 1 \qquad f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2 \qquad f(4) = \frac{4}{2} = 2$

In general $f(2m-1) = \frac{2m-1+1}{2} = 2m$

$f(2m) = \frac{\cancel{2}m}{\cancel{2}} = m$

$\therefore f(2m-1) = f(2m)$ where m is any positive int.

$\therefore f$ is not one-one function.

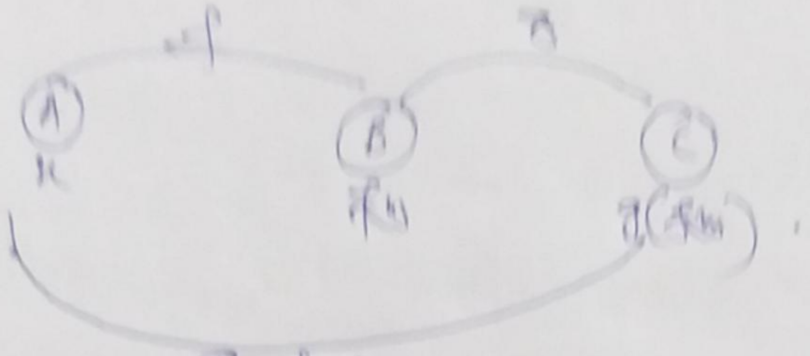
Hence f is onto as $R_f = N$ $\because$ for any $n \in N$ there is $2n \in N$ such that $f(2n) = \frac{2n}{2} = n$.

So. f is onto but not one-one., Hence f is not a bijective.

Composition of function

Definition: Let $f: A \to B$ and $g: B \to C$ be two functions. Composition of f & g, denoted by $g \circ f$, is defined as function $g \circ f: A \to C$ given by

$$g \circ f(x) = g(f(x)), \text{ for all } x \in A$$



$g \circ f(x)$.

For example let $f: \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ & $g: \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$, $g(5) = g(9) = 11$

$\therefore \quad g \circ f(1) = g(f(1)) = g(\text{Nothing}) = \text{Nothing}$
it does not exist

$g \circ f(2) = g(f(2)) = g(3) = 7 \Rightarrow g \circ f(2) = 7$

$g \circ f(3) = g(f(3)) = g(4) = 7 \Rightarrow g \circ f(3) = 7$

$g \circ f(4) = g(f(4)) = g(5) = 11 \Rightarrow g \circ f(4) = 11$

$g \circ f(5) = g(f(5)) = g(5) = 11 \Rightarrow g \circ f(5) = 11$

Invertible functions

Let $f: \{1,2,3\} \rightarrow \{a,b,c\}$ be one-one & onto function given by $f(1) = a$, $f(2) = b$ & $f(3) = c$

Now consider a function $g: \{a,b,c\} \rightarrow \{1,2,3\}$

Also suppose $X = \{1,2,3\}$ & $Y = \{a,b,c\}$

such that $g(a) = 1$, $g(b) = 2$ & $g(c) = 3$

$g \circ f(1) = g(f(1)) = g(a) = 1$

$g \circ f(2) = g(f(2)) = g(b) = 2$

$g \circ f(3) = g(f(3)) = g(c) = 3.$

$\therefore$ $g \circ f = I_X$, is identity function on X

$f \circ g = I_Y$, is identity function on Y

$f \circ g(a) = f(g(a)) = f(1) = a$

$f \circ g(b) = f(g(b)) = f(2) = b$

$f \circ g(c) = f(g(c)) = f(3) = c$ Hence proved.

$f: \{1,2,3\} \rightarrow \{a,b,c\}$ one-one & onto

$g: \{a,b,c\} \rightarrow \{1,2,3\}$ one-one & onto

Such that $g \circ f = I_X$ & $f \circ g = I_Y$

then f must be one-one & onto.

Definition:- A function $f: X \rightarrow Y$ is defined to be invertible if there exist a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called inverse of f and is denoted by $f^{-1} = g$.

Thus f is invertible, then f must be one-one and onto and conversely if f is one-one & onto, then f must be invertible.

If $f: X \rightarrow Y$, $g: Y \rightarrow Z$, $h: Z \rightarrow S$ are functions, Then

$h \circ (g \circ f) = (h \circ g) \circ f$ Sol $\rightarrow$ $h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x))) \ \forall x \in X$

$(h \circ g) \circ f(x) = h \circ g(f(x)) = h(g(f(x))) \ \forall x \in X$