

KNIT

MATRICES

By

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Matrices Definition

A matrix is a function which consists of an ordered rectangular array of numbers. The numbers in the array are called the entities or the elements of the matrix. The horizontal array of elements in the matrix is called rows, and the vertical array of elements are called the columns. If a matrix has m rows and n columns, then it is known as the matrix of order $m \times n$.

Types of Matrices

Depending upon the order and elements, matrices are classified as,

Column Matrix

A column matrix is an $m \times 1$ matrix, consisting of a single column of m elements. It is also called a column vector.

Row Matrix

A row matrix is a $1 \times m$ matrix, consisting of a single row of m elements. It is also called a row vector.

Square Matrix

A matrix which has an equal number of rows and columns. It is expressed as $m \times m$.

Diagonal Matrix

A matrix which has non-zero elements in its diagonal part running from upper left to the lower right or vice versa.

Scalar Matrix

The scalar matrix is a square matrix, which has all its diagonal elements equal and all the off-diagonal elements as zero.

Identity Matrix

A square matrix which has all its principal diagonal elements as ones and all non-diagonal elements as zeros.

Zero Matrix

A matrix whose all entries are zero. It is also called a null matrix.

Equality of Matrices

Two matrices are said to be equal if-

- (i) The order of both the matrices are the same
- (ii) Each element of one matrix is equal to the corresponding element of the other matrix.

Addition of Matrices

If $A[a_{ij}]_{m \times n}$ and $B[b_{ij}]_{m \times n}$ are two matrices of the same order then their sum $A + B$ is a matrix, and each element of that matrix is the sum of the corresponding elements. i.e. $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Consider the two matrices A & B of order 2 x 2. Then the sum is given by:

$$\begin{bmatrix} a1 & b1 \\ c1 & d1 \end{bmatrix} + \begin{bmatrix} a2 & b2 \\ c2 & d2 \end{bmatrix} = \begin{bmatrix} a1 + a2 & b1 + b2 \\ c1 + c2 & d1 + d2 \end{bmatrix}$$

Properties of Matrix Addition: If a, B and C are matrices of same order, then

(a) **Commutative Law:** $A + B = B + A$

(b) **Associative Law:** $(A + B) + C = A + (B + C)$

(c) **Identity of the Matrix:** $A + O = O + A = A$, where O is zero matrix which is additive identity of the matrix,

(d) **Additive Inverse:** $A + (-A) = 0 = (-A) + A$, where $(-A)$ is obtained by changing the sign of every element of A which is additive inverse of the matrix,

$$(e) \left. \begin{array}{l} A + B = A + C \\ B + A = C + A \end{array} \right\} \Rightarrow B = C$$

$$(f) \operatorname{tr}(A \pm B) = \operatorname{tr}(A) \pm \operatorname{tr}(B)$$

(g) If $A + B = 0 = B + A$, then B is called additive inverse of A and also A is called the additive inverse of A .

Subtraction of Matrices

If A and B are two matrices of the same order, then we

define $A - B = A + (-B)$.

Consider the two matrices A & B of order 2×2 . Then the difference is given by:

$$\begin{bmatrix} a1 & b1 \\ c1 & d1 \end{bmatrix} - \begin{bmatrix} a2 & b2 \\ c2 & d2 \end{bmatrix} =$$
$$\begin{bmatrix} a1 - a2 & b1 - b2 \\ c1 - c2 & d1 - d2 \end{bmatrix}$$

We can subtract the matrices by subtracting each element of one matrix from the corresponding element of the second matrix. i.e. $A - B = [a_{ij} - b_{ij}]_{m \times n}$

Scalar Multiplication of Matrices

If $A = [a_{ij}]_{m \times n}$ is a matrix and k any number, then

the matrix which is obtained by multiplying the elements of A by k is called the scalar multiplication of A by k and

it is denoted by kA thus if $A = [a_{ij}]_{m \times n}$

$$\text{Then } kA_{m \times n} = A_{m \times n}k = [ka_{i \times j}]$$

Properties of Scalar Multiplication: If A, B are matrices of the same order and λ, μ are any two scalars then;

$$(a) \lambda (A + B) = \lambda A + \lambda B$$

$$(b) (\lambda + \mu) A = \lambda A + \mu A$$

$$(c) \lambda (\mu A) = (\lambda \mu) A = \mu (\lambda A)$$

$$(d) (-\lambda A) = -(\lambda A) = \lambda (-A)$$

$$(e) \text{tr} (kA) = k \text{tr} (A)$$

Transpose of Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called the **transpose of matrix** A and is denoted by A^T or A' . From the definition it is obvious that if the order of A is $m \times n$, then the order of A^T becomes $n \times m$; E.g. transpose of matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2 \times 3} \text{ is } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3 \times 2} .$$

Properties of Transpose of Matrix

(i) $(A^T)^T = A$ (ii) $(A + B)^T = A^T + B^T$ (iii) $(AB)^T = B^T A^T$ (iv)
 $(kA)^T = k(A)^T$

Symmetric Matrix & Skew Symmetric Matrix

To understand if a matrix is a symmetric matrix, it is very important to know about transpose of a matrix and how to find it. If we interchange rows and columns of an $m \times n$ matrix to get an $n \times m$ matrix, the new matrix is called the transpose of the given matrix. There are two possibilities for the number of rows (m) and columns (n) of a given matrix:

- If $m = n$, the matrix is square
- If $m \neq n$, the matrix is rectangular

Symmetric matrix & Skew Symmetric Matrix

- Symmetric: $\mathbf{A}^T = \mathbf{A}$.
- Skew-symmetric: $\mathbf{A}^T = -\mathbf{A}$.
- Examples:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

skew-symmetric

EXERCISE : - 3 . 1

Question 1:

In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix (ii) The number of elements,

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$

Solution 1:

(i) In the given matrix, the number of rows is 3 and the number of columns is 4. Therefore, the order of the matrix is 3×4 .

(ii) Since the order of the matrix is 3×4 , there are $3 \times 4 = 12$ elements in it.

(iii) $a_{13} = 19, a_{21} = 35, a_{33} = -5, a_{24} = 12, a_{23} = \frac{5}{2}$

Question 2:

If a matrix has 24 elements, what are the possible order it can have? What, if it has 13 elements?

Solution 2:

We know that if a matrix is of the order $m \times n$, it has mn elements. Thus, to find all the possible orders of a matrix having 24 elements, we have to find all the ordered pairs of natural numbers whose product is 24.

The ordered pairs are: $(1, 24), (24, 1), (2, 12), (12, 2), (3, 8), (8, 3), (4, 6)$, and $(6, 4)$

Hence, the possible orders of a matrix having 24 elements are:

$1 \times 24, 24 \times 1, 2 \times 12, 12 \times 2, 3 \times 8, 8 \times 3, 4 \times 6$, and 6×4

$(1, 13)$ and $(13, 1)$ are the ordered pairs of natural numbers whose product is 13.

Hence the possible orders of matrix having 13 elements are 1×13 and 13×1 .

Question 3:

If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

Solution 3:

We know that if a matrix is of the order $m \times n$, it has mn elements. Thus, to find all the possible

orders of a matrix having 18 elements, we have to find all the ordered pairs of natural numbers whose products is 18.

The ordered pairs are: $(1,18), (18,1), (2,9), (9,2), (3,6),$ and $(6,3)$

Hence, the possible orders of a matrix having 18 elements are :

$1 \times 18, 18 \times 1, 2 \times 9, 9 \times 2, 3 \times 6,$ and 6×3

$(1,5)$ and $(5,1)$ are the ordered pairs of natural numbers whose product is 5.

Hence, the possible orders of a matrix having 5 elements are 1×5 and 5×1 .

Question 4:

Construct a 3×4 matrix, whose elements are given by

(i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

Solution 4:

In general, a 3×4 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(i) $a_{ij} = \frac{1}{2}|-3i + j|, i = 1, 2, 3, 4$

$$\therefore a_{11} = \frac{1}{2}|-3 \times 1 + 1| = \frac{1}{2}|-3 + 1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$$

$$a_{21} = \frac{1}{2}|-3 \times 2 + 1| = \frac{1}{2}|-6 + 1| = \frac{1}{2}|-5| = \frac{5}{2}$$

$$a_{31} = \frac{1}{2}|-3 \times 3 + 1| = \frac{1}{2}|-9 + 1| = \frac{1}{2}|-8| = 4$$

$$a_{12} = \frac{1}{2}|-3 \times 1 + 2| = \frac{1}{2}|-3 + 2| = \frac{1}{2}|-1| = \frac{1}{2}$$

$$a_{22} = \frac{1}{2}|-3 \times 2 + 2| = \frac{1}{2}|-6 + 2| = \frac{1}{2}|-4| = \frac{4}{2} = 2$$

$$a_{32} = \frac{1}{2}|-3 \times 3 + 2| = \frac{1}{2}|-9 + 2| = \frac{1}{2}|-7| = \frac{7}{2}$$

$$a_{13} = \frac{1}{2}|-3 \times 1 + 3| = \frac{1}{2}|-3 + 3| = 0$$

$$a_{23} = \frac{1}{2}|-3 \times 2 + 3| = \frac{1}{2}|-6 + 3| = \frac{1}{2}|-3| = \frac{3}{2}$$

$$a_{33} = \frac{1}{2}|-3 \times 3 + 3| = \frac{1}{2}|-9 + 3| = \frac{1}{2}|-6| = \frac{6}{2} = 3$$

$$a_{14} = \frac{1}{2}|-3 \times 1 + 4| = \frac{1}{2}|-3 + 4| = \frac{1}{2}|1| = \frac{1}{2}$$

$$a_{24} = \frac{1}{2} |-3 \times 2 + 4| = \frac{1}{2} |-6 + 4| = \frac{1}{2} |-2| = \frac{2}{2} = 1$$

$$a_{34} = \frac{1}{2} |-3 \times 3 + 4| = \frac{1}{2} |-9 + 4| = \frac{1}{2} |-5| = \frac{5}{2}$$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

(ii) $a_{ij} = 2i - j$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$

$$\therefore a_{11} = 2 \times 1 - 1 = 2 - 1 = 1$$

$$a_{21} = 2 \times 2 - 1 = 4 - 1 = 3$$

$$a_{31} = 2 \times 3 - 1 = 6 - 1 = 5$$

$$\therefore a_{12} = 2 \times 1 - 2 = 2 - 2 = 0$$

$$a_{22} = 2 \times 2 - 2 = 4 - 2 = 2$$

$$a_{32} = 2 \times 3 - 2 = 6 - 2 = 4$$

$$\therefore a_{13} = 2 \times 1 - 3 = 2 - 3 = -1$$

$$a_{23} = 2 \times 2 - 3 = 4 - 3 = 1$$

$$a_{33} = 2 \times 3 - 3 = 6 - 3 = 3$$

$$\therefore a_{14} = 2 \times 1 - 4 = 2 - 4 = -2$$

$$a_{24} = 2 \times 2 - 4 = 4 - 4 = 0$$

$$a_{34} = 2 \times 3 - 4 = 6 - 4 = 2$$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

Question 5:

Find the value of x , y and z from the following equation:

$$(i) \begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix} \quad (ii) \begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix} \quad (iii) \begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

Solution 5:

$$(i) \begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$$

As the given matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$$x = 1, y = 4 \text{ and } z = 3$$

$$(ii) \begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$$

As the given matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$$x + y = 6, xy = 8, 5 + z = 5$$

$$\text{Now, } 5 + z = 5 \Rightarrow z = 0$$

We know that:

$$(x - y)^2 = (x + y)^2 - 4xy$$

$$\Rightarrow (x - y)^2 = 36 - 32 = 4$$

$$\Rightarrow x - y = \pm 2$$

Now, when $x - y = 2$ and $x + y = 6$, we get $x = 4$ and $y = 2$

When $x - y = -2$ and $x + y = 6$, we get $x = 2$ and $y = 4$

$\therefore x = 4, y = 2$, and $z = 0$ or $x = 2, y = 4$, and $z = 0$

$$(iii) \begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

As the two matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$$x + y + z = 9 \quad \dots\dots(1)$$

$$x + z = 5 \quad \dots\dots(2)$$

$$y + z = 7 \quad \dots\dots(3)$$

From (1) and (2), we have:

$$y + 5 = 9$$

$$\Rightarrow y = 4$$

Then, from (3), we have:

$$4 + z = 7$$

$$\Rightarrow z = 3$$

$$\therefore x + z = 5$$

$$\Rightarrow x = 2$$

$$\therefore x = 2, y = 4, \text{ and } z = 3$$

Question 6:

Find the value of a , b , c , and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

Solution 6:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

As the two matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$$a - b = -1 \quad \dots(1)$$

$$2a - b = 0 \quad \dots(2)$$

$$2a + c = 5 \quad \dots(3)$$

$$3c + d = 13 \quad \dots(4)$$

From (2), we have:

$$b = 2a$$

Then, from (1), we have:

$$a - 2a = -1$$

$$\Rightarrow a = 1$$

$$\Rightarrow b = 2$$

Now, from (3), we have:

$$2 \times 1 + c = 5$$

$$\Rightarrow c = 3$$

From (4) we have:

$$3 \times 3 + d = 13$$

$$\Rightarrow 9 + d = 13 \Rightarrow d = 4$$

$$\therefore a = 1, b = 2, c = 3, \text{ and } d = 4$$

EXERCISE: - 3.2

Question 1:

$$\text{Let } A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$$

Find each of the following

- (i) $A + B$ (ii) $A - B$ (iii) $3A - C$ (iv) AB (v) BA

Solution 1:

$$(i) A + B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$$

$$(ii) A - B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-3 \\ 3-(-2) & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$$

$$(iii) 3A - C = 3 \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} \\ = \begin{bmatrix} 3 \times 2 & 3 \times 4 \\ 3 \times 3 & 3 \times 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} \\ = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} \\ = \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix} \\ = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$$

- (iv) Matrix A has 2 columns. This number is equal to the number of rows in matrix B.
Therefore, AB is defined as:

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2(1)+4(-2) & 2(3)+4(5) \\ 3(1)+2(-2) & 3(3)+2(5) \end{bmatrix} \\ = \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$$

- (v) matrix B has 2 columns. This number is equal to the number of rows in matrix A.
Therefore, BA is defined as:

$$BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ -2(2)+5(3) & -2(4)+5(2) \end{bmatrix} \\ = \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$$

Question 2:

Compute the following:

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad (ii) \begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix} \quad (iii) \begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$$

$$(iv) \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

Solution 2:

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$$

$$(ii) \begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix} \\ = \begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix} \\ = \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$$

$$(iii) \begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix} \\ = \begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix} \\ = \begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$$

$$(iv) \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix} \\ = \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (\because \sin^2 x + \cos^2 x = 1)$$

Question 3:

Compute the indicated products

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 3 \ 4]$$

$$(iii) \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$(v) \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Solution 3:

$$(i) \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} a(a)+b(b) & a(-b)+b(a) \\ -b(a)+a(b) & -b(-b)+a(a) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2 & -ab+ab \\ -ab+ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 3 \ 4] = \begin{bmatrix} 1(2) & 1(3) & 1(4) \\ 2(2) & 2(3) & 2(4) \\ 3(2) & 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)-2(2) & 1(2)-2(3) & 1(3)-2(1) \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 2-6 & 3-2 \\ 2+6 & 4+9 & 6+3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$$

$$\text{(iv)} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+3(0)+4(3) & 2(-3)+3(2)+4(0) & 2(5)+3(4)+4(5) \\ 3(1)+4(0)+5(3) & 3(-3)+4(2)+5(0) & 3(5)+4(4)+5(5) \\ 4(1)+5(0)+6(3) & 4(-3)+5(2)+6(0) & 4(5)+5(4)+6(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & -12+10+0 & 20+20+30 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$$

$$\text{(v)} \quad \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+1(-1) & 2(0)+1(2) & 2(1)+1(1) \\ 3(1)+2(-1) & 3(0)+2(2) & 3(1)+2(1) \\ -1(1)+1(-1) & -1(0)+1(2) & -1(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 0+2 & 2+1 \\ 3-2 & 0+4 & 3+2 \\ -1-1 & 0+2 & -1+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\text{(vi)} \quad \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(2)-1(1)+3(3) & 3(-3)-1(0)+3(1) \\ -1(2)+0(1)+2(3) & -1(-3)+0(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 6-1+9 & -9-0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$$

Question 4:

$$\text{If } A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}, \text{ then}$$

Compute $(A+B)$ and $(B-C)$. Also, verify that $A+(B-C) = (A+B)-C$

Solution 4:

$$\begin{aligned} A+B &= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} \end{aligned}$$

$$B-C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{aligned} A+(B-C) &= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+(-1) & 2+(-2) & -3+0 \\ 5+4 & 0+(-1) & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (A+B)-C &= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1-(-2) & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \end{aligned}$$

Hence, we have verified that $A+(B-C) = (A+B)-C$.

Question 5:

$$\text{If } A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix} \text{ then compute } 3A - 5B$$

Solution 5:

$$\begin{aligned} 3A - 5B &= 3 \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} - 5 \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Question 6:

$$\text{Simplify } \cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$$

Solution 6:

$$\begin{aligned} &\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{bmatrix} + \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\because \cos^2 \theta + \sin^2 \theta = 1) \end{aligned}$$

Question 7:

Find X and Y , if

$$(i) \quad X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \text{ and } X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(ii) \quad 2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \text{ and } 3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$$

Solution 7:

$$(i) \quad X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \dots\dots(1)$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \dots\dots\dots(2)$$

Adding equations (1) and (2), we get:

$$2X = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\therefore X = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\text{Now, } X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 7-5 & 0-0 \\ 2-1 & 5-4 \end{bmatrix}$$

$$\therefore Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(ii) \quad 2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \quad \dots\dots(3)$$

$$3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix} \quad \dots\dots(4)$$

Multiplying equation (3) with (2), We get:

$$2(2X + 3Y) = 2 \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow 4X + 6Y = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} \quad \dots\dots(5)$$

Multiplying equation (4) with (3), we get:

$$3(3X + 2Y) = 3 \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$$

$$\Rightarrow 9X + 6Y = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix} \quad \dots\dots(6)$$

From (5) and (6), we have:

$$(4X + 6Y) - (9X + 6Y) = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix}$$

$$\Rightarrow -5X = \begin{bmatrix} 4-6 & 6-(-6) \\ 8-(-3) & 0-15 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix}$$

$$\therefore X = -\frac{1}{5} \begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$$

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

Now,

$$\Rightarrow 2 \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix} + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix} + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix}$$

$$\Rightarrow 3Y = \begin{bmatrix} 2-\frac{4}{5} & 3+\frac{24}{5} \\ 4+\frac{22}{5} & 0-6 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix}$$

Question 8:

Find X , if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

Solution 8:

$$2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow 2X + \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\therefore X = \frac{1}{2} \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

Question 9:

Find X and Y , if $2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

Solution 9:

$$2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Comparing the corresponding elements of these two matrices, we have:

$$2 + y = 5$$

$$\Rightarrow y = 3$$

$$2x + 2 = 8$$

$$\Rightarrow x = 3$$

$$\therefore x = 3 \text{ and } y = 3$$

Question 10:

Solve the equation for X , Y , Z and t if

$$2 \begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3 \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Solution 10:

$$\begin{aligned} 2 \begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} &= 3 \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} &= \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix} &= \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} \end{aligned}$$

Comparing the corresponding elements of these two matrices, we get:

$$2x + 3 = 9$$

$$\Rightarrow 2x = 6$$

$$\Rightarrow x = 3$$

$$2y = 12$$

$$\Rightarrow y = 6$$

$$2z - 3 = 15$$

$$\Rightarrow 2z = 18$$

$$\Rightarrow z = 9$$

$$2t + 6 = 18$$

$$\Rightarrow 2t = 12$$

$$\Rightarrow t = 6$$

$$\therefore x = 3, y = 6, z = 9, \text{ and } t = 6$$

Question 11:

If $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find values of x and y .

Solution 11:

$$\begin{aligned}x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 10 \\ 5 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} &= \begin{bmatrix} 10 \\ 5 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} &= \begin{bmatrix} 10 \\ 5 \end{bmatrix}\end{aligned}$$

Comparing the corresponding elements of these two matrices, we get:

$$2x - y = 10 \quad \text{and} \quad 3x + y = 5$$

Adding these two equations, we have:

$$5x = 15$$

$$\Rightarrow x = 3$$

$$\text{Now, } 3x + y = 5$$

$$\Rightarrow y = 5 - 3x$$

$$\Rightarrow y = 5 - 9 = -4$$

$$\therefore x = 3 \text{ and } y = -4$$

Question 12:

$$\text{Given } 3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}, \text{ find the values of } x, y, z \text{ and } w.$$

Solution 12:

$$\begin{aligned}3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} &= \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}\end{aligned}$$

Comparing the corresponding elements of these two matrices, we get:

$$3x = x + 4$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

$$3x = 6 + x + y$$

$$\Rightarrow 2y = 6 + x = 6 + 2 = 8$$

$$\Rightarrow y = 4$$

$$3w = 2w + 3$$

$$\Rightarrow w = 3$$

$$3z = -1 + z + w$$

$$\Rightarrow 2z = -1 + w = -1 + 3 = 2$$

$$\Rightarrow z = 1$$

$$\therefore x = 2, y = 4, z = 1, \text{ and } w = 3$$

Question 13:

$$\text{If } F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ show that } F(x)F(y) = F(x+y).$$

Solution 13:

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(x)F(y)$$

$$= \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y + 0 & -\cos x \sin y - \sin x \cos y + 0 & 0 \\ \sin x \cos y + \cos x \sin y + 0 & -\sin x \sin y + \cos x \cos y + 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= F(x+y)$$

$$\therefore F(x)F(y) = F(x+y)$$

Question 14:

Show that

$$(i) \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution 14:

$$(i) \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5(2) - 1(3) & 5(1) - 1(4) \\ 6(2) + 7(3) & 6(1) + 7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 10 - 3 & 5 - 4 \\ 12 + 21 & 6 + 28 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2(5) + 1(6) & 2(-1) + 1(7) \\ 3(5) + 4(6) & 3(-1) + 4(7) \end{bmatrix}$$

$$= \begin{bmatrix} 10 + 6 & -2 + 7 \\ 15 + 24 & -3 + 28 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1(-1) + 2(0) + 3(2) & 1(1) + 2(-1) + 3(3) & 1(0) + 2(1) + 3(4) \\ 0(-1) + 1(0) + 0(2) & 0(1) + 1(-1) + 0(3) & 0(0) + 1(1) + 0(4) \\ 1(-1) + 1(0) + 0(2) & 1(1) + 1(-1) + 0(3) & 1(0) + 1(1) + 0(4) \end{bmatrix}$$

$$\begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1(1)+1(0)+0(1) & -1(2)+1(-1)+0(1) & -1(3)+1(0)+0(0) \\ 0(1)+(-1)(0)+1(1) & 0(2)+(-1)(1)+1(1) & 0(3)+(-1)(0)+1(0) \\ 2(1)+3(0)+4(1) & 2(2)+3(1)+4(1) & 2(3)+3(0)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Question 15:

Find $A^2 - 5A + 6I$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solution 15:

We have $A^2 = A \times A$

$$A^2 = AA \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2(2)+0(2)+1(1) & 2(0)+0(1)+1(-1) & 2(1)+0(3)+1(0) \\ 2(2)+1(2)+3(1) & 2(0)+1(1)+3(-1) & 2(1)+1(3)+3(0) \\ 1(2)+(-1)(2)+0(1) & 1(0)+(-1)(1)+0(-1) & 1(1)+(-1)(3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2 - 5A + 6I$$

$$\begin{aligned}
&= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1+5 & -2-0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} -5+6 & -1+0 & -3+0 \\ -1+0 & -7+6 & -10+0 \\ -5+0 & 4+0 & -2+6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}
\end{aligned}$$

Question 16:

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^2 - 6A + 7I = 0$

Solution 16:

$$\begin{aligned}
A^2 &= AA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}
\end{aligned}$$

Now $A^2 = A^2 \cdot A$

$$\begin{aligned} &= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} \end{aligned}$$

$\therefore A^3 - 6A^2 + 7A + 21$

$$\begin{aligned} &= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 21+7+2 & 0+0+0 & 34+14+0 \\ 12+0+0 & 8+14+2 & 23+7+0 \\ 34+14+0 & 0+0+0 & 55+21+2 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} \\ &= \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

$\therefore A^3 - 6A^2 + 7A + 21 = O$

Question 17:

If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$

Solution 17:

$$A^2 = A \cdot A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3(3)+(-2)(4) & 3(-2)+(-2)(-2) \\ 4(3)+(-2)(4) & 4(-2)+(-2)(-2) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$$\text{Now } A^2 = kA - 2I$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = k \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

Comparing the corresponding elements, we have:

$$3k - 2 = 1$$

$$\Rightarrow 3k = 3$$

$$\Rightarrow k = 1$$

Thus, the value of K is 1.

Question 18:

If $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Solution 18:

On the L.H.S.

$$I + A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \quad \dots\dots(1)$$

On the R.H.S.

$$\begin{aligned} & (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha + \sin \alpha \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \tan \frac{\alpha}{2} \\ -\cos \alpha \tan \frac{\alpha}{2} + \sin \alpha & \sin \alpha \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix} \quad \dots\dots\dots(2) \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} 1 - 2\sin^2 \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} - \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} & -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \left(2\cos^2 \frac{\alpha}{2} - 1 \right) \tan \frac{\alpha}{2} \\ -\left(2\cos^2 \frac{\alpha}{2} - 1 \right) \tan \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} + 1 - 2\sin^2 \frac{\alpha}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2\sin^2 \frac{\alpha}{2} + 2\sin^2 \frac{\alpha}{2} & -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} - \tan \frac{\alpha}{2} \\ -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \tan \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2\sin^2 \frac{\alpha}{2} + 1 - 2\sin^2 \frac{\alpha}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \end{aligned}$$

Thus, from (1) and (2), we get **L.H.S. = R.H.S.**