1. **Continuity of a function at a point :**

A function f(x) is said to be continuous at a point x = a if

(i) f(a) exists

(ii) $\lim_{x \to a} f(x)$ exists i.e. both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exists

and $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

(iii) $\lim_{x \to a} f(x) = f(a)$

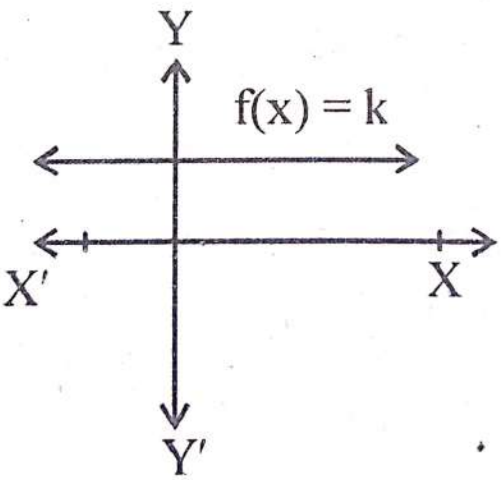
2. **Continuous functions :** A function f(x) is said to be continuous if it is continuous at each point of its domain.

(a) **Everywhere continuous function :**

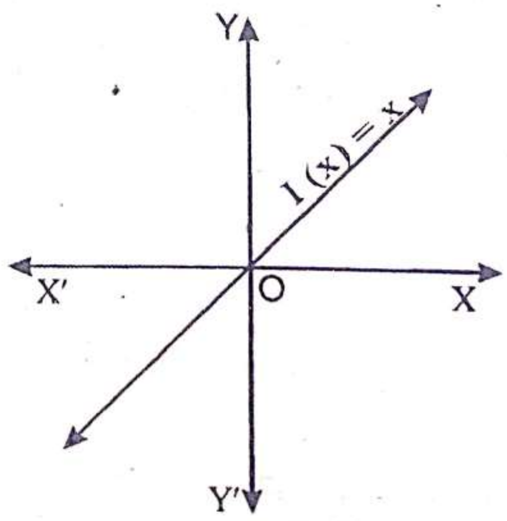
A function f(x) is said to be everywhere continuous if it is continuous on the entire real line $(-\infty, \infty)$.

(b) Listed below are some common type of functions that are continuous in their domain.

(i) **Constant function :** Every constant function f(x) = k, where k is constant is everywhere continuous.

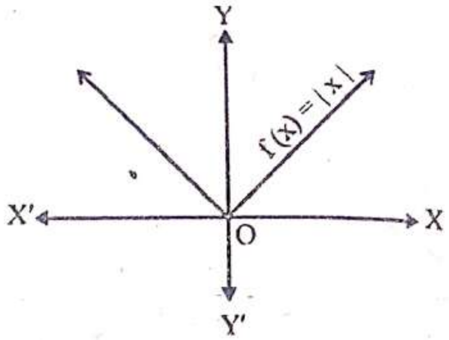


(ii) **Identity function :** The identity function I(x) is defined by I(x) = x for all $x \in R$ is everywhere continuous.



(iii) Modulus function: The modulus function is defined as

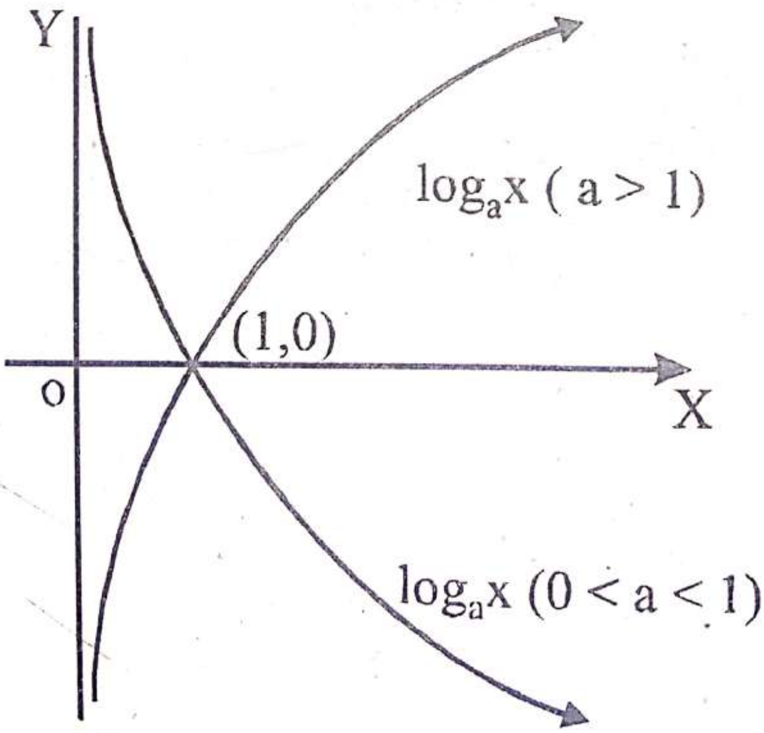
$$f(x)=|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{cases}$$



Clearly, the domain of f(x) is R and this function is everywhere continuous.

(iv) Exponential function : If a is a positive real number, other than 1, then the function f(x) defined by $f(x) = a^x$ for all $x \in R$, is called the exponential function. The domain of this function is R . It is everywhere continuous.

(v) Logarithmic function : If a is positive real number other than unity, then a function defined by $f(x) = \log_a x$ is called the logarithmic function. Clearly its domain is the set of all positive real numbers and it is continuous on its domain.



(vi) Polynomial function: A function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where $a_0, a_1, a_2, \ldots, a_n \in R$ is called a polynomial function. This function is everywhere continuous.

3. **Algebra of continuous functions:**

If f and g are two continuous functions on their common domain D, then

(i) $f+g$ is continuous on D.

(ii) $f-g$ is continuous on D.

(iii) fg is continuous on D.

(iv) αf is continuous on D where α is any real number.

(v) f/g is continuous on $D-\{x : g(x) \neq 0\}$

(vi) $1/f$ is continuous on $D-\{x : f(x) \neq 0\}$

(vii) The composition of two continuous functions is a continuous function.

(viii) If f is continuous on its domain D, then $|f|$ is also continuous on D.

4. **Discontinuous functions** : A function is said to be a discontinuous function if it is discontinuous at least one point in its domain.

The discontinuity may arise due to any of the following situations:

(i) If at least one of f(a), $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist.

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to f (a).

We classify the points of discontinuity as :

(a) Removable discontinuity:

A function f is said to have removable discontinuity at $x = a$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ but their common value is not equal to f (a).

Such a discontinuity can be removed by assigning a suitable value to the function f at $x = a$.

(b) Discontinuity of the first kind:

A function f is said to have a discontinuity of the first kind at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal.

f is said to have a discontinuity of the first kind from the left at $x = a$ if $\lim_{x \to a^-} f(x)$ exists but not equal to f(a). Discontinuity of the first kind from the right is similary defined.

(c) Discontinuity of second kind:

A function f is said to have a discontinuity of the second kind at $x = a$ if neither $\lim_{x \to a^-} f(x)$ nor $\lim_{x \to a^+} f(x)$ exists.

f is said to have a discontinuity of the second kind from the left at $x = a$ if $\lim_{x \to a^-} f(x)$ does not exist.

Similarly, if $\lim_{x \to a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at $x = a$.

Examples of some discontinuous function

(i) $f(x) = 1/x$ at $x = 0$

(ii) $f(x) = e^{1/x}$ at $x = 0$

(iii) $f(x) = \sin 1/x$, $f(x) = \cos 1/x$ at $x = 0$

(iv) $f(x) = [x]$ at every integer

(v) $f(x) = x - [x]$ at every integer

(vi) $f(x) = \tan x$, $f(x) = \sec x$; when $x = (2n + 1)\pi/2, n \in z$

(vii) $f(x) = \cot x$, $f(x) = \text{cosec } x$; when $x = n\pi, n \in z$

5. **Differentiability of a function :**

A function f (x) is said to be differentiable at a point of its domain if it has a finite derivative at that point.

Thus f(x) is differentiable at x = a

if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists finitely

$$\Rightarrow \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

⇒ Left hand derivative = Right hand derivative

Generally derivative of f (x) at x = a is denoted by f′ (a)

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

6. **Differentiable function :**

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

7. **Some standard results on differentiability :**

(i) Every polynomial function is differentiable at each $x \in R$.

(ii) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(iii) Every constant function is differentiable at each $x \in R$.

(iv) The logarithmic function is differentiable at each point in its domain.

(v) Trigonometric and inverse-trigonometric functions are differentiable in their domains.

(vi) The sum, difference, product and quotient of two differentiable functions is differentiable.

(vii) The composition of differentiable function is a differentiable function.

(viii) If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.

(ix) If f (x) and g (x) both are not differentiable at a point, then the sum function f(x) + g (x) and the product function f(x). g(x) can still be differentiable at that point.

8. **Relation between continuity and differentiability :**

(i) If a function f(x) is differentiable at a point x = a then it is continuous at x = a.

(ii) If f(x) is continuous at a point x = a, then f(x) may or may not be differentiable at there.

(iii) If f(x) is not differentiable at x = a then it may or may not be continuous at x = a.

(iv) If f(x) is not continuous at x = a, then it is not differentiable at x = a.

(v) If left hand derivative and right hand derivative of f(x) at x = a are finite (they may or may not be equal) then f(x) is continuous at x = a.

9. **Algebra of a differentiable function**

(i) If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ will be derivable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

(ii) If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $f(x)\,g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ and $g(x)$ both are not differentiable at $x = a$, then the product function $f(x)\,g(x)$ can still be differentiable at $x = a$.

(iv) If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the sum function $f(x) + g(x)$ is not differentiable at $x = a$.

(v) If $f(x)$ and $g(x)$ both are not - differentiable at $x = a$, then the sum function may be a differentiable function.
e.g. $f(x) = |x| + 1$ and $g(x) = -|x|$

(vi) If $f(x)$ is derivable at $x = a$ then it need not be true that $f'(x)$ is continuous at $x = a$.

10. **Differentiation of some standard functions :**

(i) $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

(ii) $$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

(iii) $$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

(iv) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(vi) $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

11. Theorems on differentiation :

Theorem I : Derivative of the function of the function. If 'y' is a function of 't' and 't' is a function of 'x' then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

Theorem II : Derivative of parametric equations if $x = f(t), y = \psi(t)$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Theorem III : Derivative of a function with respect to another function :

If f (x) and g (x) are two functions of a variable x, then

$$\frac{d[f(x)]}{d[g(x)]} = \frac{d}{dx}f(x) / \frac{d}{dx}g(x)$$

Theorem IV : $\frac{dy}{dx}.\frac{dx}{dy} = 1.$

12. Methods of differentiation : Working rule for finding the derivative of implicit functions

(i) Differentiate every term of $f(x, y) = 0$ with respect to x.

(ii) Collect the coefficients of $\frac{dy}{dx}$ and obtain the value of $\frac{dy}{dx}$.

13. Differentiation of logarithmic functions :

If differentiation of an expression or an equation is done by taking log on both sides, then, it is called logarithmic differentiation. This method is useful for the function having following forms.

(i) When base and power both are the functions of x i.e. the functions is of the form $[f(x)]^{g(x)}$.

$$y = [f(x)]^{g(x)}$$

$$\Rightarrow \log y = g(x) \log [f(x)]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}[g(x). \log [f(x)]]$$

$$\Rightarrow \frac{dy}{dx} = [f(x)]^{g(x)}. \left\{\frac{d}{dx}[g(x) \log f(x)]\right\}$$

(ii) If a function is the product and quotient of many simpler functions such as

$$y = \frac{f_1(x).f_2(x)...}{g_1(x).g_2(x)....}$$

We first take logarithm and then differentiate.

Note : The main point to be noted in this method that the both function must always be positive.

14. Some suitable substitutions

Functions	Substitutions
(i) $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii) $\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii) $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
(iv) $\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos 2\theta$
(v) $\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$	$x^2 = a^2\cos 2\theta$
(vi) $\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(vii) $\sqrt{\dfrac{x}{a+x}}$	$x = a\tan^2\theta$
(viii) $\sqrt{\dfrac{x}{a-x}}$	$x = a\sin^2\theta$
(ix) $\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(x) $\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

15. n^{th} derivatives of some standard functions :

(a) $\dfrac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$

(b) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$, where $m < n$

(c) $\frac{d^n}{dx^n}(\log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(d) $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(e) $\frac{d^n(a^x)}{dx^n} = a^x (\log a)^n$

(f) $\frac{d^n}{dx^n}[e^{ax} \sin(bx+c)] = r^n e^{ax} \sin(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2}$; $\phi = \tan^{-1} \frac{b}{a}$

16. MEAN VALUE THEOREMS :

(i) Rolle's Mean Value Theorem :

Let f be a real function defined in the closed interval [a, b] such that :

(i) $f(a) = f(b)$

(ii) f is continuous in the closed interval [a, b]

(iii) $f(x)$ is differentiable in the open interval (a, b)

Then there is at least one point c in the open interval (a, b) such that $f'(c) = 0$.

Note :

1. There can be more than one-such points
2. If f(x) be a polynomial function such that α and β are its zeros. That is $f(\alpha) = 0$ and $f(\beta) = 0$

   Since a polynomial function is continuous and differentiable every where on the real number line, hence the conditions of Rolle's theorem are satisfied. Then there exists γ, such that

   $f'(\gamma) = 0, \quad \alpha < \gamma < \beta$

Alternatively :

Between two roots of a polynomial equation $f(x) = 0$, there exists at least one value γ of x such that $f'(\gamma) = 0$

(ii) Lagrange's mean value theorem :

Let f be a real function, continuous on the closed interval [a, b] and differentiable in the open interval (a, b). Then there is atleast one point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks :

In the particular case where $f(a) = f(b)$.

The expression $\frac{f(b)-f(a)}{b-a}$ becomes zero. Thus when

$f(a) = f(b)$, $f'(c) = 0$ for at least one value c in (a, b).

Thus Rolle's theorem becomes a particular case of the mean value theorem.

17. **Intermediate value of theorem :**

If f is continuous in [a, b], then f assumes at least once every values between minimum and maximum values of f(x)

Thus, $a \leq x \leq b \Rightarrow m \leq f(x) \leq M$

or range of $f(x) = [m, M]$ if $x \in [a\ b]$

Application of intermediate value of theorem

Exactly one root of $f(x) = 0$ lies between the given numbers p and q if $f(p)f(q) \leq 0$. But $f(p)$ and $f(q)$ are not simultaneously zero.